A sharp height estimate for compact spacelike hypersurfaces with constant $r$-mean curvature in the Lorentz–Minkowski space and application

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Dedicated to my wife Márcia and my son Nicolás

Abstract

In this paper we obtain a sharp height estimate concerning compact spacelike hypersurfaces $Σ^n$ immersed in the $(n + 1)$-dimensional Lorentz–Minkowski space $L^{n+1}$ with some nonzero constant $r$-mean curvature, and whose boundary is contained into a spacelike hyperplane of $L^{n+1}$. Furthermore, we apply our estimate to describe the nature of the end of a complete spacelike hypersurface of $L^{n+1}$.

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1. Introduction

In the last years, the study of spacelike hypersurfaces in the Lorentz–Minkowski space $L^{n+1}$ has been of substantial interest from both the physical and mathematical aspects. From a physical point of view, that interest is motivated by their role in the study of different problems in general relativity (see, for example, [15] and [21]). From a mathematical point of view, that interest is also motivated by the fact that these hypersurfaces exhibit nice Bernstein-type properties. Actually, Calabi in [9], for $n \leq 4$, and Cheng and Yau in [11], for arbitrary $n$, showed that the only complete spacelike hypersurfaces in $L^{n+1}$ with zero mean curvature are the spacelike hyperplanes. More recently, Aiyama in [1] and Xin in [24] simultaneously and independently characterized the spacelike hyperplanes as the only complete constant mean curvature spacelike hypersurfaces in $L^{n+1}$ having the image of its Gauss map contained in a geodesic ball in the $n$-dimensional hyperbolic space $H^n$ (see also [22] for a weaker first version of this result given by Palmer). Furthermore,
related with the compact case, Alías and Malacarne in [4] showed that the only compact spacelike hypersurfaces having constant higher order mean curvature and spherical boundary are the hyperplanar balls with zero higher order mean curvature, and the hyperbolic caps with nonzero constant higher order mean curvature (see also [5] for the case of constant mean curvature and [6] for the case of constant scalar curvature).

In this paper we deal with compact spacelike hypersurfaces immersed in \( L^{n+1} \). Specifically, supposing the boundary contained into a spacelike hyperplane, we obtain a height estimate concerning these hypersurfaces in the case that they have some nonzero constant \( r \)-mean curvature. We prove the result below.

**Theorem (Theorem 4.2).** Let \( \psi : \Sigma^n \to L^{n+1} \) be a compact spacelike hypersurface whose boundary is contained in \( [0] \times \mathbb{R}^n \). Suppose that the \( r \)th mean curvature \( H_r \) is constant and positive (after an appropriated choice of the Gauss map \( N \) of \( \Sigma \), if \( r \) is odd). If the hyperbolic image of \( \Sigma^n \) is contained in a geodesic ball of center \( e_{n+1} \in \mathbb{H}^n \) and radius \( \varrho > 0 \), then the height \( h \) of \( \Sigma^n \) satisfies the inequality

\[
|h| \leq \frac{\cosh(\varrho) - 1}{H_r^{1/r}}.
\]

As an application of this estimate, we obtain a version of the theorems of Aiyama in [1] and Xin in [24], for the case of constant higher order mean curvature. More precisely, we prove the following

**Theorem (Theorem 5.1).** Let \( \psi : \Sigma^n \to L^{n+1} \) be a complete spacelike hypersurface with one end. Suppose that the \( r \)th mean curvature \( H_r \neq 0 \) is constant. If the hyperbolic image of \( \Sigma \) is contained into a geodesic ball of \( \mathbb{H}^{n+1} \), then its end is not divergent.

To prove our results, we consider the Lorentz–Minkowski space as a **Robertson–Walker** spacetime (cf. Section 2) and then we apply the technique used by Hoffman, de Lira and Rosenberg in [16], where they obtained such estimate concerning compact vertical graphs in a product \( M^2 \times \mathbb{R} \), being \( M^2 \) a Riemannian surface (in fact, Rosenberg in [23] already obtained this type of estimate concerning such graphs in the Riemannian space forms; see also Xu Cheng and Rosenberg in [12] where they work in a Riemannian product \( M^m \times \mathbb{R} \)).

We observe that the hyperbolic caps are examples of spacelike hypersurfaces that realize our estimate for the height function (see Remark 4.5); so, our estimate is sharp. In this sense, Theorem 4.2 is an answer to the conjecture made by López in [18], where (using another framework) he obtained another sharp estimate for the height of compact spacelike surfaces \( \Sigma^2 \) immersed into the 3-dimensional Lorentz–Minkowski space \( \mathbb{L}^3 \). Finally, let us remark that hyperbolic caps also show that if we fix the \( r \)th mean curvature \( H_r \), there exist spacelike graphs immersed in the Lorentz–Minkowski space with constant \( r \)-mean curvature \( H_r \) and with arbitrary height. This not occurs in Euclidean setting.

2. Preliminaries

Let \( M^n \) be a connected, \( n \)-dimensional oriented Riemannian manifold, \( I \subset \mathbb{R} \) an interval and \( f : I \to \mathbb{R} \) a positive smooth function. In the product differentiable manifold \( \overline{M}^{n+1} = I \times M^n \), let \( \pi_I \) and \( \pi_M \) denote the projections onto the factors \( I \) and \( M \), respectively.

A particular class of Lorentzian manifolds (**spacetimes**) is the one obtained by furnishing \( \overline{M} \) with the metric

\[
(v, w)_p = -((\pi_I)_* v, (\pi_I)_* w) + ((f \circ \pi_I)(p))^2((\pi_M)_* v, (\pi_M)_* w),
\]

for all \( p \in \overline{M} \) and all \( v, w \in T_p \overline{M} \). Such a space is called (following the terminology introduced in [7]) a **Generalized Robertson–Walker** (GRW) spacetime, and in what follows we shall write \( \overline{M}^{n+1} = -I \times f M^n \) to denote it. In particular, when \( M^n \) has constant sectional curvature then \( -I \times f M^n \) is classically called a **Robertson–Walker** (RW) spacetime (cf. [21]).

We recall that a tangent vector field \( K \) on a spacetime \( \overline{M}^{n+1} \) is said to be conformal if the Lie derivative with respect to \( K \) of the metric \( \langle \cdot, \cdot \rangle \) of \( \overline{M}^{n+1} \) satisfies:

\[
\mathcal{L}_K \langle \cdot, \cdot \rangle = 2\phi \langle \cdot, \cdot \rangle,
\]
Moreover, in the 3-dimensional case, denoting by $A$ the shape operator $\Sigma n$ point in $\Sigma n$, the Gaussian curvature of the spacelike surface $\psi$ is given by $\calg$, which, as usual, is also denoted by $\langle , \rangle$. In this setting, $\nabla$ stands for the Levi-Civita connection of $\Sigma n$.

Let $A$ be the shape operator of $\Sigma n$ in $\cal$ associated to the choice of an orientation $N$ of $\Sigma n$. Associated with the shape operator $A$ there are $n$ algebraic invariants, which are the elementary symmetric functions $S_r$ of its principal curvatures $\kappa_1, \ldots, \kappa_n$, given by

$$S_r = S_r(\kappa_1, \ldots, \kappa_n) = \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \cdots \kappa_{i_r}, \quad 1 \leq r \leq n.$$ 

The $r$-mean curvature $H_r$ of the spacelike hypersurface $\Sigma n$ is then defined by

$$\binom{n}{r} H_r = (-1)^r S_r(\kappa_1, \ldots, \kappa_n) = S_r(-\kappa_1, \ldots, -\kappa_n).$$

In particular, when $r = 1$,

$$H_1 = -\frac{1}{n} \sum_{i=1}^n \kappa_i = -\frac{1}{n} \text{tr}(A) = H$$

is the mean curvature of $\Sigma n$, which is the main extrinsic curvature of the hypersurface. The choice of the sign $(-1)^r$ in our definition of $H_r$ is motivated by the fact that in that case the mean curvature vector is given by $\vec{H} = H N$. Therefore, $H(p) > 0$ at a point $p \in \Sigma$ if and only if $\vec{H}(p)$ is in the same time-orientation as $N(p)$ (in the sense that $\langle \vec{H}, N \rangle_p < 0$).

When $r = 2$, $H_2$ defines a geometric quantity which is related to the (intrinsic) scalar curvature $R$ of the hypersurface. For instance, when the ambient spacetime $\cal$ has constant sectional curvature $\tilde{\kappa}$, it follows from the Gauss’ equation that

$$R = n(n-1)(\tilde{\kappa} - H_2).$$

Moreover, in the 3-dimensional case, denoting by $K_\Sigma$ the Gaussian curvature of the spacelike surface $\psi : \Sigma^2 \rightarrow \cal$, we have that

$$K_\Sigma = \tilde{\kappa} - H_2.$$

From the ideas of Montiel and Ros concerning Lemma 1 in [19] and their use of Gardin inequalities (cf. [13]), and taking into account our sign convention in the definition of the $r$-mean curvature, we derive the following result (see also [10], Proposition 2.3).

**Lemma 2.1.** Let $\Sigma n$ be a spacelike hypersurface immersed in a spacetime $\cal$. Suppose that there exists an elliptic point in $\Sigma n$. If $H_r$ is positive on $\Sigma n$, we have that the same holds for $H_k$, $k = 1, \ldots, r - 1$. Moreover,

$$H_{k-1} \geq H_{k-1}^{(k-1)/k} \quad \text{and} \quad H \geq H_1^{1/k},$$

$k = 1, \ldots, r$. If $k \geq 2$, in the above inequalities the equality happens only at umbilical points.

Here, by an elliptic point in a spacelike hypersurface $\Sigma n$ we mean a point $p_0 \in \Sigma n$ where all principal curvatures $\kappa_i(p_0)$ are negative with respect an appropriate choice of the Gauss map $N$ of $\Sigma n$. 

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3. The Newton transformations

In this section, we will introduce the corresponding Newton transformations

\[ P_r : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma), \quad 0 \leq r \leq n, \]

which arise from the shape operator \( A \). According to our definition of the \( r \)-mean curvatures, the Newton transformations are given by

\[ P_r = \binom{n}{r} H_r I + \binom{n}{r-1} H_{r-1} A + \cdots + \binom{n}{1} H_1 A^{r-1} + A^r, \]

where \( I \) denotes the identity in \( \mathcal{X}(\Sigma) \), or, inductively,

\[ P_0 = I \quad \text{and} \quad P_r = \binom{n}{r} H_r I + A \circ P_{r-1}. \]

Observe that the characteristic polynomial of \( A \) can be written in terms of the \( H_r \) as

\[ \det(tI - A) = \sum_{r=0}^{n} \binom{n}{r} H_r t^{n-r}, \]

where \( H_0 = 1 \). By the Cayley–Hamilton theorem, this implies that \( P_n = 0 \). Besides, we have the following properties of \( P_r \) (cf. [2]).

1. If \( \{e_1, \ldots, e_n\} \) is a local orthonormal frame on \( \Sigma \) which diagonalizes \( A \), i.e., \( Ae_i = \kappa_i e_i, i = 1, \ldots, n \), then it also diagonalizes each \( P_r \), and \( P_r e_i = \lambda_{i,r} e_i \) with

\[ \lambda_{i,r} = (-1)^r \sum_{i_1 < \cdots < i_r, i_j \neq i} \kappa_{i_1} \cdots \kappa_{i_r} = \sum_{i_1 < \cdots < i_r, i_j \neq i} (-\kappa_{i_1}) \cdots (-\kappa_{i_r}). \]

2. For each \( 1 \leq r \leq n - 1 \),

\[ \text{tr}(P_r) = (r + 1) \binom{n}{r+1} H_r, \]

\[ \text{tr}(A \circ P_r) = -(r + 1) \binom{n}{r+1} H_{r+1} \]

and

\[ \text{tr}(A^2 \circ P_r) = \binom{n}{r+1} (n H_1 H_{r+1} - (n-r-1) H_{r+2}). \]

3. For every \( V \in \mathcal{X}(\Sigma) \) and for each \( 1 \leq r \leq n - 1 \),

\[ \text{tr}(P_r \circ \nabla_V A) = -\binom{n}{r+1} \langle \nabla H_{r+1}, V \rangle. \]

4. When the ambient spacetime has constant sectional curvature, the Newton transformations \( P_r \) are divergence free, that is,

\[ \text{div}_\Sigma(P_r) = \text{tr}(V \to (\nabla_V P_r)V) = 0. \]

Associated to each Newton transformation \( P_r \) one has the second order linear differential operator \( L_r : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma) \), given by

\[ L_r(f) = \text{tr}(P_r \text{ Hess } f). \]
Therefore, for \( f, g \in \mathcal{D}(\Sigma) \), it follows from the properties of the Hessian of functions that
\[
L_r(fg) = f L_r(g) + g L_r(f) + 2(P_r \nabla f, \nabla g).
\]

Observe that
\[
L_r(f) = \text{tr}(P_r \text{Hess } f) = \sum_{i=1}^{n} \langle P_r(\nabla e_i \nabla f), e_i \rangle
\]
where \( \{e_1, \ldots, e_n\} \) is a local orthonormal frame on \( \Sigma \). Moreover, when the spacetime ambient has constant sectional curvature, we also have
\[
\text{div}(P_r(\nabla f)) = \sum_{i=1}^{n} \langle (\nabla e_i P_r)(\nabla f), e_i \rangle + \sum_{i=1}^{n} \langle P_r(\nabla e_i \nabla f), e_i \rangle
\]
\[
= \langle \text{div } P_r, \nabla f \rangle + L_r(f) = L_r(f).
\]

Consequently, we conclude that the operator \( L_r \) is elliptic if and only if \( P_r \) is positive definite. We observe that \( L_0 = \Delta \) is always elliptic. The next lemma gives a geometric condition which guarantees the ellipticity of \( L_1 \).

**Lemma 3.1.** (See Lemma 3.2 of [3].) Let \( \Sigma \) be a spacelike hypersurface immersed into a GRW spacetime. If \( H_2 > 0 \) on \( \Sigma \), then \( L_1 \) is elliptic or, equivalently, \( P_1 \) is positive definite (for a appropriate choice of the Gauss map \( N \)).

**Proof.** Simple observe that by Cauchy–Schwarz inequality we have \( H^2 \geq H_2 \), and \( H \) does not vanish on \( \Sigma \). By choosing the appropriate Gauss map \( N \), we may assume that \( H > 0 \). Recall that \( H_2 \) does not depend on the chosen \( N \).

Since
\[
n^2H^2 = \sum_{i=1}^{n} \kappa_i^2 + n(n-1)H_2 > \kappa_j^2,
\]
for every \( j = 1, \ldots, n \), then \( \lambda_{j,1} = nH + \kappa_j > 0 \) for all \( j \) and, consequently, \( P_1 \) is positive definite. \( \square \)

When \( r \geq 2 \), the following lemma give us sufficient conditions to guarantee the ellipticity of \( L_r \) (for the proof see [12], Proposition 3.2; see also [8], Proposition 3.2).

**Lemma 3.2.** (See Lemma 3.3 of [3].) Let \( \Sigma \) be a spacelike hypersurface immersed into a GRW spacetime. If there exists an elliptic point of \( \Sigma \), with respect to an appropriate choice of the Gauss map \( N \), and \( H_{r+1} > 0 \) on \( \Sigma \), for \( 2 \leq r < n-1 \), then for all \( 1 \leq k \leq r \) the operator \( L_k \) is elliptic or, equivalently, \( P_k \) is positive definite (for a appropriate choice of the Gauss map \( N \), if \( k \) is odd).

To close this section, we present the analytical framework that we will use to obtain our results. The following formulæ were obtained by L.J. Alías and A.G. Colares (cf. [3]). Here, and for the sake of completeness, we present more direct and specific proofs.

**Proposition 3.3.** Let \( \psi : \Sigma^n \to \mathcal{M}^{n+1} = -I \times_{f} M^n \) be a spacelike hypersurface immersed into a GRW spacetime, with Gauss map \( N \). Let \( h = \pi_1 \circ \psi \) denote the height function of \( \Sigma \). Then, for every \( r = 0, \ldots, n-1 \) we have
\[
L_r h = -(\log f)'(h) \left( (r+1) \left( \begin{array}{c} n \\ r+1 \end{array} \right) H_r + \langle P_r(\nabla h), \nabla h \rangle \right) - \langle N, \partial_r \rangle (r+1) \left( \begin{array}{c} n \\ r+1 \end{array} \right) H_{r+1}.
\]
Moreover, if the spacetime \( \mathcal{M}^{n+1} \) is flat, then
\[
L_r \langle N, K \rangle = \left( \begin{array}{c} n \\ r+1 \end{array} \right) \left( \langle \nabla H_{r+1}, K \rangle + (r+1) f'(h) H_{r+1} \right) + \langle N, K \rangle \text{tr}(A^2 \circ P_r),
\]
where \( K = f \partial_r \).
Proof. One has
\[ \nabla h = \nabla (\pi_N) = (\nabla \pi_v)^\top = -\partial_t^\top \]
where \( \nabla \) denotes the gradient with respect to the metric of the ambient space and \( X^\top \) the tangential component of a vector field \( X \in X(M) \) in \( \Sigma \). Now fix \( p \in M, v \in T_pM \) and let \( A \) denote the Weingarten map with respect to \( N \). Write \( v = w - \langle v, \partial_t \rangle \partial_t \), so that \( w \in T_pM \) is tangent to the fiber of \( M \) passing through \( p \). By repeated use of the formulas of item (2) of Proposition 7.35 of [21], we get
\[
\nabla_v \partial_t = \nabla_w \partial_t - \langle v, \partial_t \rangle \nabla_{\partial_t} \partial_t = \nabla_w \partial_t
\]
\[
= (\log f)'w = (\log f)'(v + \langle v, \partial_t \rangle \partial_t).
\]
Thus,
\[
\nabla_v \nabla h = \nabla_v \nabla h + (Av, \nabla h)N
\]
\[
= \nabla_v (-\partial_t - \langle N, \partial_t \rangle N) + (Av, \nabla h)N
\]
\[
= -(\log f)'w - \langle \langle N, \partial_t \rangle \rangle N + \langle N, \partial_t \rangle Av + (Av, \nabla h)N
\]
\[
= -(\log f)'w + \langle \langle Av, \partial_t \rangle \rangle N + \langle N, \partial_t \rangle Av + (Av, \nabla h)N
\]
\[
= -(\log f)'w - (\log f)'(v, \partial_t)N + \langle N, \partial_t \rangle Av
\]
\[
= -(\log f)'(v + \langle v, \partial_t \rangle \nabla h)\n\]
\[
+ (N, \partial_t)Av
\]
\[
= -(\log f)'(v + \langle v, \nabla h \rangle \nabla h) + (N, \partial_t)Av.
\]

Now, by fixing \( p \in \Sigma \) and an orthonormal frame \( \{e_i\} \) at \( T_p \Sigma \), one gets
\[
L_r h = \text{tr}(P_r, \text{Hess } h) = \sum_{i=1}^n \langle \nabla_{e_i} \nabla h, P_r e_i \rangle
\]
\[
= \sum_{i=1}^n \langle -(\log f)'(e_i + \langle e_i, \nabla h \rangle \nabla h) + \langle N, \partial_t \rangle Ae_i, P_r e_i \rangle
\]
\[
= -(\log f)'\left[ \text{tr}(P_r) + \langle P_r(\nabla h), \nabla h \rangle \right] + \langle N, \partial_t \rangle \text{tr}(A \circ P_r).
\]
Therefore, the formula of \( L_r h \) follows from the expressions of the traces of \( P_r \) and \( A \circ P_r \).

To prove the formula for \( L_r \langle N, K \rangle \), suppose that our spacetime \( \tilde{M}^{n+1} \) is flat. Since \( \nabla_V K = f'(h)V \) for all \( V \in X(\Sigma) \), we easily see that
\[
\nabla \langle N, K \rangle = -A(K^\top).
\]
Thus, for all \( V \in X(\Sigma) \),
\[
\nabla_V (\nabla \langle N, K \rangle) = -A(V) = A(K^\top) - A(V).
\]
Then, by Codazzi equation and since \( \nabla h = -\partial_t^\top \), we get
\[
\nabla_V (\nabla \langle N, K \rangle) = -A(K^\top) - f'(h)A(V) + \langle N, K \rangle A^2(V).
\]
Therefore, using the general fact that (cf. [2])
\[
\text{tr}(\nabla A \circ P_r) = \text{tr}(P_r \circ \nabla A) = -\left( \frac{n}{r + 1} \right) \langle \nabla H_{r+1}, V \rangle,
\]
we obtain the desired formula of \( L_r \langle N, K \rangle \). \( \square \)
4. Height estimate in the Lorentz–Minkowski space

Let $\mathbb{L}^{n+1}$ denote the $(n+1)$-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$, endowed with the Lorentz metric

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i - v_{n+1} w_{n+1},$$

for all $v, w \in \mathbb{R}^{n+1}$. We note that $\mathbb{L}^{n+1}$ can be regarded as a RW spacetime; more precisely:

$$\mathbb{L}^{n+1} = -\mathbb{R} \times \mathbb{R}^n.$$

Considering the RW model of $\mathbb{L}^{n+1}$, we will deal with a compact spacelike hypersurface $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ whose boundary $\partial \Sigma$ is a $(n-1)$-dimensional closed submanifold embedded in the horizontal hyperplane $\Pi = \{0\} \times \mathbb{R}^n$ (for simplification, we will just say that the boundary of $\Sigma$ is contained in $\Pi$). In what follows, we will subvert the compositions with the isometry $\Phi$ between the RW and the canonical models of $\mathbb{L}^{n+1}$ which is characterized by

$$(\Phi_x)(\partial_t) = e_{n+1} = (0, \ldots, 0, 1) \quad \text{and} \quad \Phi(\{0\} \times \mathbb{R}^n) = \{x \in \mathbb{L}^{n+1}; x_{n+1} = 0\}.$$  

We note that the timelike unit normal vector field $N \in \mathcal{X}^\perp(\Sigma)$ can be regarded as a map $N : \Sigma^n \to \mathbb{H}^n$, where $\mathbb{H}^n$ denotes the $n$-dimensional hyperbolic space, that is

$$\mathbb{H}^n = \{x \in \mathbb{L}^{n+1}; \langle x, x \rangle = -1, x_{n+1} \geq 1\}.$$  

In this setting, the image $N(\Sigma)$ will be called the hyperbolic image of $\Sigma$. We will establish an estimate for the height function concerning the case of constant higher order mean curvature. For that, we need the following lemma due to L.J. Alías and J.M. Malacarne (cf. [4], Lemma 1).

**Lemma 4.1. (Existence of an elliptic point.)** Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a spacelike immersion of a compact hypersurface with boundary contained into a hyperplane $\Pi = a^{-1}$, $a$ being a unit timelike vector in the same time-orientation as the Gauss map $N$ of $\Sigma^n$. Then, for an appropriated choice of $N$, there exists an elliptic point $p_0 \in \Sigma^n$, unless the hypersurface is entirely contained in the hyperplane $\Pi$.

Finally, since the existence of an elliptic point implies the ellipticity of the operator $L_r$, now we are in the position to enunciate and prove our main result.

**Theorem 4.2.** Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a compact spacelike hypersurface whose boundary is contained in $\{0\} \times \mathbb{R}^n$. Suppose that $H_r \neq 0$ is constant. If the hyperbolic image of $\Sigma^n$ is contained in a geodesic ball of center $e_{n+1} \in \mathbb{H}^n$ and radius $\varrho > 0$, then the height $h$ of $\Sigma^n$ satisfies the inequality

$$|h| \leq \frac{\cosh(\varrho) - 1}{|H_r|^{1/r}}.$$  

**Proof.** Since (after an appropriated choice of the Gauss map $N$ of $\Sigma$, if $r$ is odd) the Lemma 4.1 guarantees the existence of an elliptic point in $\Sigma$, we can suppose that $H_r > 0$. We then consider the two possible cases.

(a) Suppose $N$ in the same time-orientation of $\partial_t$ (i.e., $\langle N, \partial_t \rangle \leq -1$).

In this case, from Proposition 3.3 we get

$$L_{r-1} h = -r \left(\begin{array}{c} n \\ r \end{array}\right) H_r \langle N, \partial_t \rangle > 0.$$  

Thus, since we are in the position to apply Lemma 3.2 (or Lemma 3.1 for the case $r = 2$), we have the ellipticity of $L_{r-1}$ and, consequently, $h \leq 0$ on $\Sigma$.

Now, we define on $\Sigma$ the function

$$\varphi = ch - \langle N, \partial_t \rangle.$$  

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where $c$ is a negative constant. Since the hypothesis on the hyperbolic image of $\Sigma$ amounts to
\[1 \leq -\langle N, \partial_t \rangle \leq \cosh(\varrho),\]
we have that $\varphi|_{\partial \Sigma} \leq \cosh(\varrho)$. On the other hand, using once more Proposition 3.3 we have that
\[L_{r-1} \varphi = -\langle N, \partial_t \rangle \left( r \frac{n}{r} c H_r + \text{tr}(A^2 \circ P_{r-1}) \right).\]
On the other hand, we have in the case $r = 1$ (by the Cauchy–Schwarz inequality)
\[HH_r - H_{r+1} = H^2 - H_2 \geq 0.\]
In the case $r > 1$, we know from Lemma 2.1 that
\[H_r - 1 \geq H^{(r-1)/r} > 0,\]
and also that
\[H \geq H_r^{-1/(r-1)}.\]
Moreover, from Newton inequalities (cf. [14], Theorem 51, p. 52, Theorem 144, p. 104; see also [10], Proposition 2.3), we have that $H^2 - H_r H_{r+1} \geq 0$. Thus,
\[H_r + 1 \leq \frac{H^2}{H_r - 1}.\]
Then, from these above inequalities, we obtain
\[HH_r - H_{r+1} \geq \frac{H_r}{H_r - 1}(HH_r - H_r) = \frac{H_r}{H_r - 1}(HH_r - H_r^{(r-1)}) = H_r(1 - H_r^{1/(r-1)}) \geq 0.\]
Therefore,
\[\text{tr}(A^2 \circ P_{r-1}) = \left( \frac{n}{r} \right) (n H H_r - (n - r) H_{r+1}) \geq r \left( \frac{n}{r} \right) H_r^{(r+1)/r}.\]
Consequently, by taking
\[c = -H_r^{1/r}\]
in the definition of the function $\varphi$, we get that $L_{r-1} \varphi \geq 0$ on $\Sigma$. Then, since Lemma 3.2 (or Lemma 3.1, for the case $r = 2$) guarantees the ellipticity of $L_{r-1}$, we conclude from the maximum principle that $\varphi \leq \cosh(\varrho)$ on $\Sigma$. Therefore,
\[0 \geq h \geq \frac{\cosh(\varrho) - 1}{H_{1/r}} = \frac{\cosh(\varrho) - 1}{H_r^{1/r}}.\]
(b) Suppose now that $N$ in the opposite time-orientation of $\partial_t$ (i.e., $\langle N, \partial_t \rangle \geq 1$).
In this case, we have $L_{r-1} \varphi < 0$ and, consequently, $h \geq 0$ on $\Sigma$. So, we define on $\Sigma$ the function
\[\varphi = ch + \langle N, \partial_t \rangle,\]
where $c$ is a positive constant. From this point, by taking
\[c = H_r^{1/r}\]
and working in a similar way as in the item (a) we conclude that
\[0 \leq h \leq \frac{\cosh(\varrho) - 1}{H_r^{1/r}}.\]

In particular, for the 3-dimensional Lorentz–Minkowski space, we obtain the following estimate of the height function $h$ of a compact spacelike surface $\Sigma^2$. 

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Corollary 4.3. Let \( \psi : \Sigma^2 \to \mathbb{L}^3 \) be a compact spacelike surface whose boundary is contained in \( \{0\} \times \mathbb{R}^2 \). Suppose that the Gaussian curvature \( K_\Sigma \neq 0 \) is constant. If the hyperbolic image of \( \Sigma^2 \) is contained in a geodesic ball of center \( e_3 \in \mathbb{H}^2 \) and radius \( \rho > 0 \), then the height \( h \) of \( \Sigma^2 \) is such that

\[
|h| \leq \frac{\cosh(\rho) - 1}{(-K_\Sigma)^{1/2}}.
\]

Observing that the translation \( \Phi_{t_0} \) by an arbitrary real parameter \( t_0 \), that is,

\[
\Phi_{t_0} : \{t\} \times \mathbb{R}^n \mapsto \{t + t_0\} \times \mathbb{R}^n,
\]

is an isometry of \( \mathbb{L}^{n+1} \), we get the following result.

Corollary 4.4. Let \( \psi : \Sigma^n \to \mathbb{L}^{n+1} \) be a compact spacelike hypersurface whose boundary is contained in \( \{t_0\} \times \mathbb{R}^n \). Suppose that \( H_r \neq 0 \) is constant. If the hyperbolic image of \( \Sigma^n \) is contained in a geodesic ball of center \( e_{n+1} \in \mathbb{H}^n \) and radius \( \rho > 0 \), then

\[
\Sigma \subset [t_0, t_0 + C] \times \mathbb{R}^n, \quad \text{or} \quad \Sigma \subset [t_0 - C, t_0] \times \mathbb{R}^n,
\]

where \( C = (\cosh(\rho) - 1)|H_r|^{-1/r} \).

Remark 4.5. Fixed a positive constant \( \lambda \), we easily verify that the hyperbolic cap

\[
\Sigma_\lambda^n = \left\{ x \in \mathbb{L}^{n+1}; \langle x, x \rangle = -\lambda^2, \lambda \leq x_{n+1} \leq \sqrt{1 + \lambda^2} \right\}
\]

is an example of spacelike hypersurface of the Lorentz–Minkowski space \( \mathbb{L}^{n+1} \) which has \( r \)-mean curvature

\[
H_r = \frac{1}{\lambda^r} > 0,
\]

for each \( 1 \leq r \leq n \) (if we choose the Gauss map \( N \) in the same time-orientation of \( e_{n+1} \), for the case \( r \) odd). Moreover, we also easily verify that the hyperbolic image of \( \Sigma_\lambda^n \) is contained in the geodesic ball of center \( e_{n+1} \in \mathbb{H}^n \) and radius \( \rho = \cosh^{-1} \sqrt{1 + \lambda^2} \).

Therefore, since the height of such hyperbolic cap is

\[
h = \sqrt{1 + \lambda^2} - \lambda = \frac{\cosh(\rho) - 1}{H_r^{1/r}},
\]

we conclude that our estimate for the height function is sharp.

5. An application

In this section we deal with complete spacelike hypersurface \( \psi : \Sigma^n \to \mathbb{L}^{n+1} \) with one end \( N^n \), that is, a hypersurface \( \Sigma^n \) that we can regarded as

\[
\Sigma^n = \Sigma^n_t \cup N^n,
\]

where \( \Sigma^n_t \) is a compact hypersurface with boundary contained into a hyperplane \( \Pi_t = \{t\} \times \mathbb{R}^n \) and \( N^n \) is diffeomorphic to the cylinder \( \{t, \infty\} \times S^{n-1} \).

Given a complete spacelike hypersurface \( \Sigma^n = \Sigma^n_t \cup N^n \) with one end, we say that its end \( N^n \) is divergent if, considering \( N^n \) with coordinates \( p = (s, q) \in \{t, \infty\} \times S^{n-1} \), we have that

\[
\lim_{s \to \infty} h(p) = \infty,
\]

where \( h \) denotes the height function of the end \( N^n \).

Now, we present the following application of our height estimate, which is a version of the theorems of Aiyama in [1] and Xin in [24] for the case of constant higher order mean curvature.
**Theorem 5.1.** Let $\psi: \Sigma^n \to \mathbb{H}^{n+1}$ be a complete spacelike hypersurface with one end. Suppose that the $r$th mean curvature $H_r \neq 0$ is constant. If the hyperbolic image of $\Sigma^n$ is contained into a geodesic ball of $\mathbb{H}^n$, then its end is not divergent.

**Proof.** Suppose, by contradiction, that the end $N^n$ of $\Sigma^n = \Sigma^n_t \cup N^n$ is divergent. Then, since $\Sigma^n_t$ is a compact hypersurface with boundary contained into a hyperplane $\Pi_t$, we have from Corollary 4.4 that (for example)

$$\Sigma_t^n \subset [t - C, t],$$

where $C = (\cosh(q) - 1)H_r^{-1/r}$ and $q$ is the radius of the geodesic ball of $\mathbb{H}^n$ that contains the hyperbolic image of $\Sigma^n$. Now, using the assumption that the end $N^n$ of $\Sigma^n$ is divergent, we can intersect $\Sigma^n$ by the hyperplane $\Pi_{t+C}$ to obtain a compact hypersurface $\Sigma_{t+C}^n$ with constant $r$-mean curvature $H_r$, with boundary contained into the hyperplane $\Pi_{t+C}$, and whose height is strictly greater than $C$. At this point, observe that (in order to apply our estimate) there is no loss of generality to consider the intersection of $\Sigma^n$ with $\Pi_{t+C}$ being transversal. Therefore, we get a contradiction with respect to our estimate for the height function of a compact spacelike hypersurface. □

**Remark 5.2.** Let us observe that, given a positive real constant $\lambda$,

$$\Sigma^n = \{ x \in \mathbb{H}^{n+1}; \langle x, x \rangle = -\lambda^2, x_{n+1} \geq \lambda \}$$

is an example of complete spacelike hypersurface with constant (nonzero) $r$-mean curvature and with one end, which is divergent. On the other hand, the hyperbolic image of $\Sigma^n$ is exactly $\mathbb{H}^n$.

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